Stokes Flow through a Thin Screen with Patterned Holes

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Three-dimensional Stokes flow through a thin screen which has a regular array of holes with two orthogonal axes of symmetry has been studied. The governing equation is simplified by the Roscoe potential and solved by an efficient eigenfunction expansion and collocation method. The resistance is found for circular and square holes in square arrays, and circular and hexagonal holes in triangular arrays.

Introduction

The flow through a screen or sieve is very important in industrial and biological filtration processes and membrane separations. Due to the small size of the holes and the low velocity, the flow can be adequately described by the Stokes equation. The problem is difficult even in the Stokes limit, and we shall confine our study to thin screens, that is, the depth of the holes is small compared to their mean diameter.

The Stokes flow through a single circular hole on a thin wall was first solved by Sampson (1891), by separating the Stokes equation in ellipsoidal coordinates. His theoretical value of pressure drop has been substantiated by experiments (Happel and Brenner, 1965). The relation between Stokes flow and a potential was noted by Oberbeck (1876) in his study of the settling of ellipsoids. Later, Roscoe (1949) used electrostatic potential to solve the Stokes flow through an elliptic hole. Hasimoto (1958) applied Roscoe's method to the Stokes flow through two-dimensional slits on a thin wall. Using conformal transforms in potential theory. Hasimoto was able to express the viscous flow results in closed form.

The situation is different when the screen has an infinite array of holes which is the more common case. Since the flow is three-dimensional, complex transforms do not apply. In this article, we shall use the Roscoe potential and an eigenfunction expansion and collocation method to solve the problem. Theoretical predictions for the pressure drop through screens with various patterned holes will be presented. An added bonus to our endeavor is the solution to the potential flow through a complementary screen with the locations of holes and walls interchanged.

Basic Equations

Consider constant density Stokes flow through a thin screen with infinite number of holes. The governing equations are:

$$\nabla p = \nabla^2 q \tag{1}$$

$$\nabla \cdot q = 0 \tag{2}$$

Here, q is the velocity vector normalized by the uniform velocity U far from the screen; all lengths are normalized by a length scale L; and p is the pressure normalized by $\mu U/L$ where μ = viscosity. Let the plane of the screen be represented by $x=0$ in Cartesian axes (x, y, z). Due to symmetry, we need to consider only the downstream ($x>0$) half space. Let $q=(u, v, w)$. The boundary conditions are:

$$q(\infty,y,z) = (1,0,0) \tag{3}$$

$$q(0,y,z) = (0,0,0) \text{ on wall} \tag{4}$$

$$\frac{\partial u}{\partial x}(0,y,z) = v(0,y,z) = w(0,y,z) = 0 \text{ in hole} \tag{5}$$

Roscoe introduces the potential ϕ such that

$$q = (\phi,0,0) - x\nabla\phi \tag{6}$$

Then, Eqs. 1-5 simplify to:

$$\nabla^2\phi = 0 \tag{7}$$

$$\phi(\infty,y,z) = \frac{P}{4}x + 1 \tag{8}$$

$$\phi(0,y,z) = 0 \text{ on wall} \tag{9}$$

$$\phi_x(0,y,z) = 0 \text{ in hole} \tag{10}$$

The pressure is

$$p = -2\phi_x \tag{11}$$

and we have set $p=0$ in hole and $p \to -P/2$ as $x \to \infty$. The total pressure drop, P, is to be determined. Note that if $P/4$ is regarded as velocity at infinity, Eqs. 7-10 define the potential flow through a complementary screen where the locations of wall and holes are interchanged.

Holes with Symmetry Arranged in a Rectangular Pattern

Consider the cases where all the holes have the same shape and are symmetrical with respect to the y and z directions. The holes are arranged in an array such that the period in the z direction is $2L$ and the period in the y direction is $2aL$. Due to symmetry, we need to consider only the rectangular stream tube of normalized dimensions 2 by $2a$. The x axis is at the center of the stream tube and pierces the center of a hole (Figure 1). Define

$$\psi(x,y,z) \equiv \phi(x,y,z) - \frac{P}{4}x - 1 \tag{12}$$

Then, in terms of ψ

$$\nabla^2\psi = 0 \tag{13}$$

$$\psi(\infty,y,z) = 0 \tag{14}$$

$$\psi(0,y,z) = -1 \text{ on wall} \tag{15}$$

$$\psi_x(0,y,z) = -\frac{P}{4} \text{ in hole} \tag{16}$$

Additional boundary conditions are that the normal velocities vanish on the surface of the stream tube or

$$\psi_y(x,\pm a,z) = 0 \tag{17}$$

$$\psi_z(x,y,\pm 1) = 0 \tag{18}$$

Noting that ψ is even and periodic in both y and z we can set:

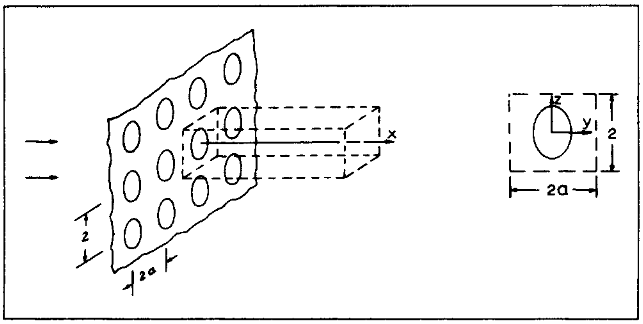

Figure 1. Flow through a screen.

$$\psi = \sum_{\substack{m,n=0 \\ m+n\neq 0}}^{\infty}\sum A_{mn}e^{-\alpha x}\cos(m\pi y/a)\cos(n\pi z) \tag{19}$$

where $\alpha \equiv \sqrt{(m/a)^2+n^2}\pi$. Equation 19 satisfies Eqs. 13, 14, 17 and 18. The unknown constants A_{mn} are to be determined from the mixed boundary conditions (Eqs. 15, 16):

$$\sum_{\substack{m,n=0 \\ m+n\neq 0}}^{\infty}\sum A_{mn}\cos(m\pi y/a)\cos(n\pi z) = -1 \text{ on wall} \tag{20}$$

$$\sum_{\substack{m,n=0 \\ m+n\neq 0}}^{\infty}\sum A_{mn}\sqrt{\left(\frac{m}{a}\right)^2+n^2}\cos(m\pi y/a)\cos(n\pi z) - D = 0 \text{ in hole} \tag{21}$$

where $D \equiv P/4\pi$. Equations 20 and 21 are solved by collocation on a finite number of points. We truncate n to N and m to $M=\text{Int}(aN)$. There are $(N+1)$ $(M+1)$ unknowns $D, A_{01}, A_{10}, A_{11}, \ldots, A_{MN}$. We choose evenly spaced points in the first quadrant, $M+1$ points in y direction and $N+1$ points in z direction. Depending on the locations, either Eqs. 20 or 21 is satisfied. In most cases, convergence to 1% error can be achieved by increasing N to about 30. The velocity distribution is then

$$(u,v,w) = (\psi+1, -x\psi_y, -x\psi_z) \tag{22}$$

The pressure is

$$p = -2(\psi_x + \pi D) \tag{23}$$

Holes in a Square Pattern

Let the holes be circular with a normalized radius $c(\leq 1)$ and placed in a square-array two units apart ($a=1$). Now, if the holes are far apart ($c \to 0$) interactions among holes would be minimal and we expect Sampson's (1891) pressure drop for a single circular hole to be valid. Normalized by $\mu U/L$, Sampson's result is:

$$P = \frac{12}{c^3} \tag{24}$$

or

$$D = \frac{3}{\pi c^3},\ c \to 0 \tag{25}$$

The value of D (pressure drop normalized by $\mu U/4\pi L$) for general c is found by solving Eqs. 20 and 21. Since the geometry is symmetrical with respect to y, z, only the points in half of the first quadrant need to be collocated:

$$z = j/N \quad j=0,i \quad y=i/N \quad i=0,N \tag{26}$$

There are $(N+1)$ $(N+2)/2$ unknowns, since A_{mn} is symmetric. The result is shown in Figure 2. We see that when $c<0.4$, the

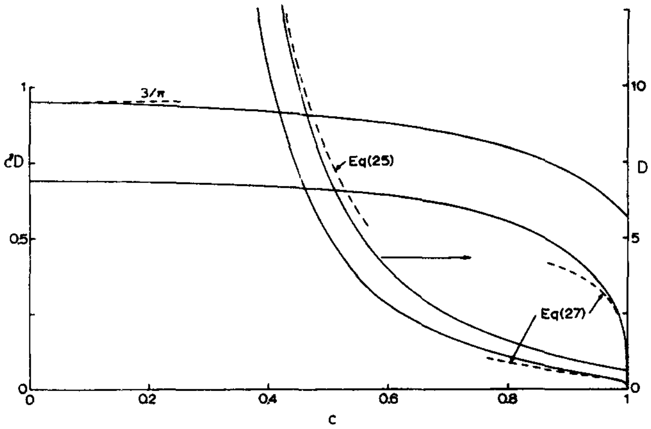

Figure 2. Pressure drop for holes in a square array.

Top curves are for circular holes, bottom curves are for square holes. Dashed lines are asymptotic results.

error for D is less than 5% between our computed results and Eq. 25. The value of D decreases from infinity to 0.565 when c increases to 1 (circular holes touching). We also plotted c^3D which is bounded as $c \to 0$.

Another important case is a screen with square holes in a square array. Let $2c$ be the length of one side of the hole. The computed result for the pressure drop is also shown in Figure 2. As $c \to 0$ and $D \to \infty$, c^3D is expected to be a constant which depends only on the shape of the hole. We find $c^3D \to 0.70$, which is the resistance for a single square hole for which no analytic solution is yet known. On the other limit when $c \to 1$ (no wall), $D \to 0$. For c close to 1, the screen is a square netting. A theoretical asymptotic form can be obtained as follows. Hasimoto (1958) derived the theoretical resistance of the flow through a screen of parallel strips. Our square netting is similar to two of Hasimoto's screens placed at right angles, provided that c is close to 1. In terms of our variables, the result is:

$$D = \frac{1}{|\ln \cos(\pi c/2)|}, \quad c \to 1 \tag{27}$$

To illustrate the flow, the velocity u and the pressure p along the centerline of the square hole is plotted in Figure 3 for $c = 0.5$. The velocity approaches 1 as $x \to \pm\infty$. We see the effect

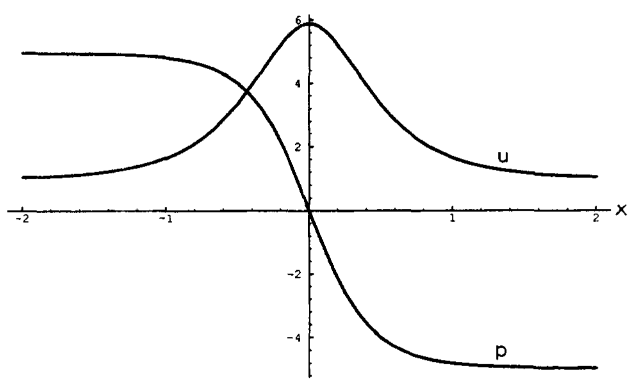

Figure 3. Pressure p and velocity u along the axis, square hole in square array, $c = 0.5$.

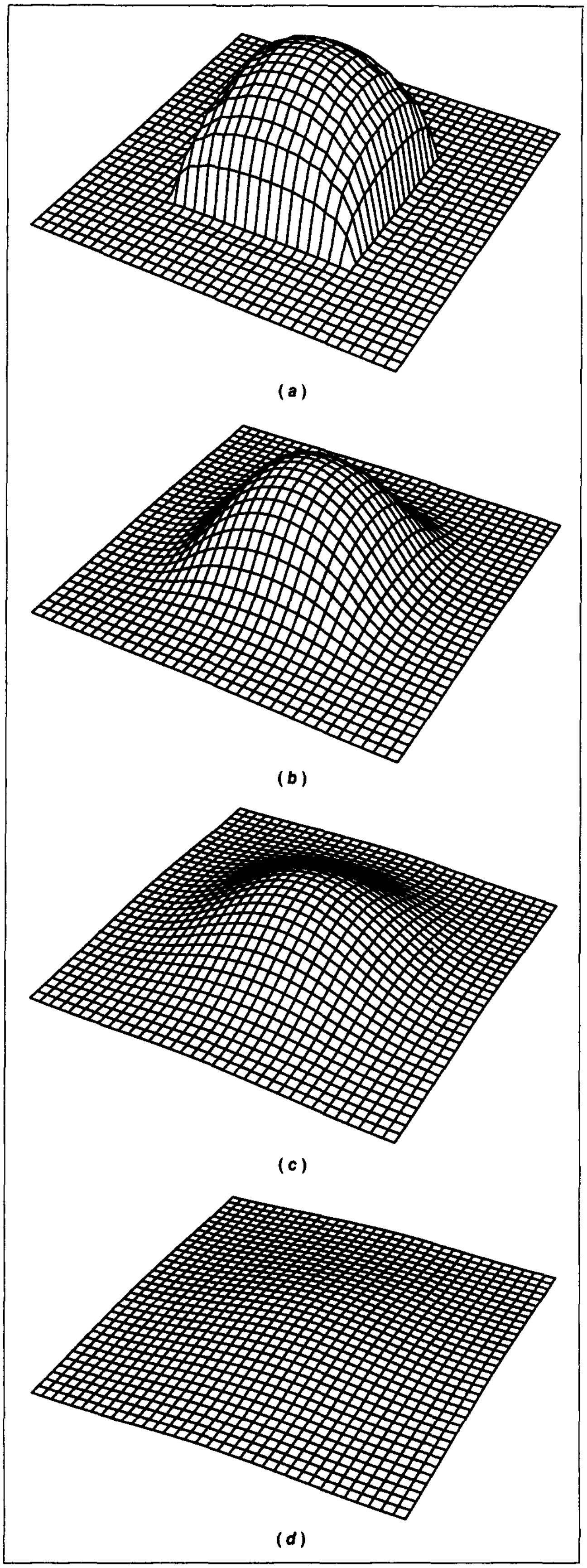

Figure 4. Axial velocity profiles, square hole in square array: (a) $x = 0$; (b) $x = 0.2$; (c) $x = 0.5$; (d) $x = 1$.

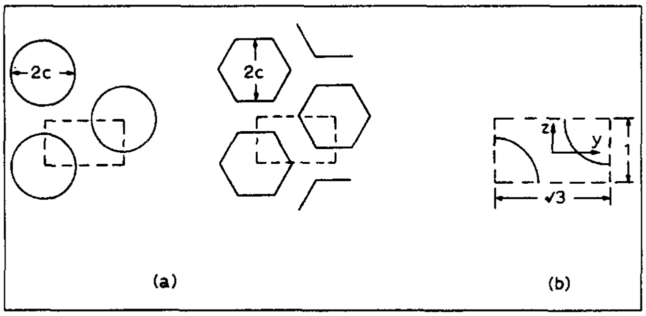

Figure 5. (a) Circular and hexagonal holes in a triangular array; (b) coordinate axes.

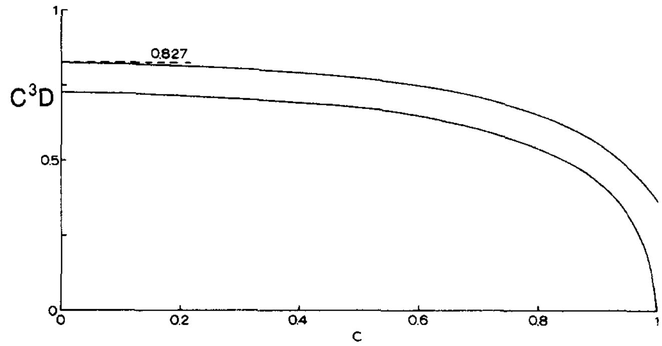

Figure 6. Pressure drop for holes in a triangular array.
Top curve is for circular holes, bottom curve is for hexagonal holes.

of the hole is limited to about one unit from the hole, which is a characteristic of Stokes flows. The axial velocity distribution is shown in Figure 4 for various location x. For $x>0.5$, the geometry of the hole is no longer discernible from the wake.

Holes in a Triangular Pattern

Circular or hexagonal holes arranged in a triangular pattern are also common (Figure 5a). Figure 5b shows a cross section of the stream tube bounded by symmetry planes. Let Cartesian axes be located at the center as shown. The solution would have polar symmetry with respect to y, z. Equations 13-16 are still the same, but Eqs. 17 and 18 are replaced by:

$$\psi_y(x, \pm\sqrt{3}/2, z) = 0 \tag{28}$$

$$\psi_z\left(x, y, \pm\frac{1}{2}\right) = 0 \tag{29}$$

We set

$$\psi = \sum_{\substack{m,n=0 \\ m+n\neq 0}}^{\infty}\sum A_{mn}e^{-\gamma x}\cos(2m\pi y/\sqrt{3})\cos(2n\pi z)$$

$$+\sum_{m,n=0}^{\infty}\sum B_{mn}e^{-\beta x}\sin[(2m+1)\pi y/\sqrt{3}]\sin[(2n+1)\pi z] \tag{30}$$

where $\gamma = 2\pi\sqrt{m^2/3+n^2}$, $\beta = \pi\sqrt{(2m+1)^2/3+(2n+1)^2}$. Equation 30 satisfies all the conditions except for the mixed boundary conditions at $x=0$. We truncate the series such that n ranges from 0 to N and m ranges from 0 to $M=\mathrm{Int}(\sqrt{3}N)$. There are $2(N+1)(M+1)$ unknowns in D, A_{mn}, B_{mn}. Due to polar symmetry, the collocation points are placed in the $y>0$ half only.

$$y = \frac{\sqrt{3}}{2(2M-1)} + \frac{\sqrt{3}i}{M}\frac{(M-1)}{(2M-1)},\ i=0 \text{ to } M \tag{31}$$

$$z = -0.5 + \frac{j}{2N+1},\ j=0 \text{ to } 2N+1 \tag{32}$$

Our results for circular holes of radius c is plotted in Figure 6. When $c = 1$, the touching case, $c^3D=0.363$ is much smaller than that of the square pattern. The corresponding Sampson value for $c\rightarrow 0$ is:

$$c^3D = \frac{3\sqrt{3}}{2\pi} = 0.827 \tag{33}$$

The results for hexagonal holes are also shown in Figure 6, where $2c$ is the distance between opposite edges of the hexagonal hole. When $c\rightarrow 1$ the screen is a hexagonal netting. As $c\rightarrow 0$, the value of $c^3D\rightarrow 0.725$. Thus, the pressure drop across a single hexagonal hole is:

$$\Delta p' = \frac{4\pi(0.725)}{2\sqrt{3}}\frac{\mu Q}{(c')^3} = 2.63\frac{\mu Q}{(c')^3} \tag{34}$$

Here, Q is the flow rate and c' is the dimensional half width of the hole.

Discussions and Conclusions

Figures 2 and 6 are useful when the nominal dimensions for a specific geometry, such as the width of a square hole and its periodicity, are used in estimating the pressure difference. To compare different-shaped holes we use the equivalent radius r (radius of a circular hole of same size, similar to hydraulic radius, normalized by the period of the pattern) and (area) void fraction f of the holes. The result is shown in Figure 7. It is evident that global properties, such as r and f alone, could not have determined the resistance of the screen. From Figure 7 we can infer the following:

- In the limit of $f\rightarrow 0$, $r^3D \sim b_1 - b_2f$. The constant b_1 depends on both the geometric shape and the pattern of the holes; the constant b_2 seems to depend only slightly on the pattern, if at all. Since f is proportional to r^2, $D \sim b_1/r^3 - b_3/r$ for small r.
- For the same r, f and pattern, circular holes give the least resistance. The difference between circular and hexagonal holes is much smaller than that between circular and square holes.

We also extrapolated the resistances of single holes in an infinite plane (Table 1). It is possible to compute theoretical values for holes of any shape using the present method.

The solution to truly three-dimensional flows (not axisymmetric) has always been difficult to obtain. The absence of a single stream function and the lack of a suitable intrinsic co-

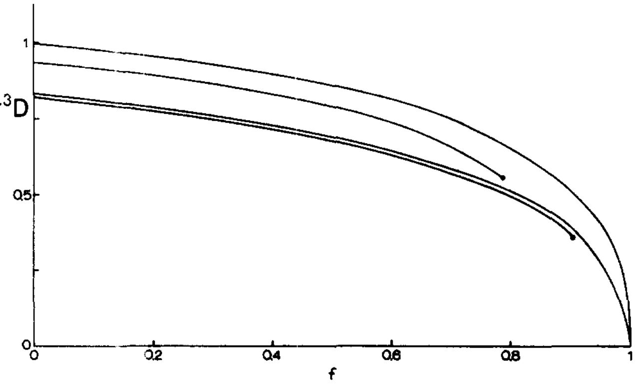

Figure 7. Pressure drop using equivalent radius r and void fraction f.

Top to bottom: square holes in square array, circular holes in square array, hexagonal holes in triangular array, circular holes in triangular array. Circular dot denote touching case.

ordinate system are major obstacles to an analytic solution. Even for linear problems such as potential or Stokes flow, the powerful complex transform method fails. On the other hand, direct numerical integration of the governing equations are possible, but unsuitable. This is because not only corners and "infinity" need to be compromised; the amount of computations would be prohibitive.

For Stokes flow through a screen, we are fortunate to lower the order of the governing equations through the Roscoe potential. For holes with two axes of symmetry, double eigenfunction expansion further simplifies the problem to satisfying mixed boundary conditions on a finite planar region. Methods such as boundary integrals or Fourier inversion or least-squares integrals could have been used, but we find that point match collocation is most versatile and easiest to implement. If direct numerical integration is used on the problem, the amount of computations would be more than the cube of those used in this article.

Table 1. Limiting Values of the Pressure Drop for a Single Hole in an Infinite Plane

	Circle (Sampson, 1891)	Hexagon	Square
c'	radius	half min. width	half length of side
$\dfrac{\Delta p'}{Q\mu/(c')^3}$	3	2.63	2.20
$\dfrac{\Delta p'}{Q\mu/r^3}$	3	3.04	3.16

Our results for the pressure drop through screens with arrays of holes should be highly useful to researchers in filtration technology. These results could be applied to industrial perforated metal plates which are thin screens with patterned holes (Warring, 1981; Cheremisinoff and Azbel, 1983). In comparison to other filter media, perforated metal plates have the advantages of more exact size control, easier cleaning, easier discharge and longer life. It is hoped that this article would elicit experimental confirmation of the theoretical results.

Notation

a = aspect ratio of pattern
A = constant coefficient
b_1, b_2, b_3 = constants
B = constant coefficient
c = normalized half width of hole
D = $P/4\pi$
f = void fraction
i, j = integer index
L = half vertical period
m, n = integer index
M, N = integer
p = normalized pressure
P = normalized total pressure drop
q = velocity vector
Q = flow rate
r = normalized equivalent radius
U = velocity at infinity
u, v, w = velocity components
x, y, z = Cartesian axes

Greek letters

α, β, γ = constants related to m, n
∇ = del operator
μ = viscosity
ϕ = Roscoe potential
ψ_1 = potential defined by Eq. 12

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